International J.Math. Combin. Vol.4(2013), 15-30

The Jordan θ-Centralizers of Semiprime Gamma Rings with Involution

M.F.Hoque and Nizhum Rahman

(Department of Mathematics, Pabna University of Science and Technology, Bangladesh)

E-mail: fazlul_math@yahoo.co.in, nl.nizhum@gmail.com

Abstract: Let M be a 2-torsion free semiprime Γ-ring with involution I satisfying a certain assumption and let $\theta : M \to M$ be an endomorpism of M. We prove that if $T : M \to M$ is an additive mapping such that $2T(x\alpha x) = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$ holds for all $x \in M$ and $\alpha \in \Gamma$, then T is a Jordan θ-centralizer with involution.

Key Words: Semiprime Γ-ring, involution, semiprime Γ-ring with involution, centralizer, θ-centralizer, Jordan θ-centralizer.

AMS(2010): 16N60, 16W25, 16U80

§1. Introduction

An extensive generalized concept of classical ring set forth the notion of a gamma ring theory. As an emerging field of research, the research work of classical ring theory to the gamma ring theory has been drawn interest of many algebraists and prominent mathematicians over the world to determine many basic properties of gamma ring and to enrich the world of algebra. The different researchers on this field have been doing a significant contributions to this field from its inception. In recent years, a large number of researchers are engaged to increase the efficacy of the results of gamma ring theory over the world.

Let M and Γ be additive abelian groups. If there exists a mapping $(x, \alpha, y) \to x\alpha y$ of $M \times \Gamma \times M \to M$, which satisfies the conditions

(i) $x\alpha y \in M$;

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)z = x\alpha z + x\beta z$, $x\alpha(y + z) = x\alpha y + x\alpha z$.

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ-ring.

Every ring M is a Γ-ring with $M = \Gamma$. However a Γ-ring need not be a ring. Gamma rings, more general than rings, were introduced by Nobusawa[11]. Bernes[2] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa.

Let M be a Γ-ring. Then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subset U (U\Gamma M \subset U)$. If U is both a left and a right ideal , then we say U is an ideal of M.

[1]Received July 15, 2013, Accepted November 16, 2013.

Suppose again that M is a Γ-ring. Then M is said to be a 2-torsion free if $2x$=0 implies x=0 for all $x \in M$. An ideal P_1 of a Γ-ring M is said to be prime if for any ideals A and B of M, $A\Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal P_2 of a Γ-ring M is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A Γ-ring M is said to be prime if $a\Gamma M\Gamma b$=(0) with $a, b \in M$, implies a=0 or b=0 and semiprime if $a\Gamma M\Gamma a$=(0) with $a \in M$ implies a=0. Furthermore, M is said to be commutative Γ-ring if $x\alpha y$=$y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover,the set $Z(M)$ =$\{x \in M : x\alpha y = y\alpha x$ for all $\alpha \in \Gamma, y \in M\}$ is called the centre of the Γ-ring M.

If M is a Γ-ring, then $[x, y]_\alpha$=$x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x, y \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:

$$[x\alpha y, z]_\beta = [x, z]_\beta\alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta \text{ and } [x, y\alpha z]_\beta = [x, y]_\beta\alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. We consider the following assumption:

$$(A)x\alpha y\beta z = x\beta y\alpha z \text{ for all } x, y, z \in M\text{and}\alpha, \beta \in \Gamma.$$

According to the assumption (A), the above two identifies reduce to

$$[x\alpha y, z]_\beta\text{=}[x, z]_\beta\alpha y + x\alpha[y, z]_\beta \text{ and } [x, y\alpha z]_\beta\text{=}[x, y]_\beta\alpha z + y\alpha[x, z]_\beta,$$

which we extensively used.

An additive mapping $T : M \to M$ is a left(right) centralizer if $T(x\alpha y)$=$T(x)\alpha y(T(x\alpha y) = x\alpha T(y))$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. A centralizer is an additive mapping which is both a left and a right centralizer. For any fixed $a \in M$ and $\alpha \in \Gamma$, the mapping $T(x) = a\alpha x$ is a left centralizer and $T(x) = x\alpha a$ is a right centralizer. We shall restrict our attention on left centralizer, since all results of right centralizers are the same as left centralizers. An additive mapping $D : M \to M$ is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all $x, y \in M$, and $\alpha \in \Gamma$ and is called a Jordan derivation if $D(x\alpha x) = D(x)\alpha x + x\alpha D(x)$ for all $x \in M$ and $\alpha \in \Gamma$.

An additive mapping $T : M \to M$ is Jordan left(right) centralizer if

$$T(x\alpha x) = T(x)\alpha x(T(x\alpha x) = x\alpha T(x))$$

for all $x \in M$ and $\alpha \in \Gamma$. Every left centralizer is a Jordan left centralizer but the converse is not ingeneral true.

An additive mappings $T : M \to M$ is called a Jordan centralizer if $T(x\alpha y + y\alpha x) = T(x)\alpha y + y\alpha T(x)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Every centralizer is a Jordan centralizer but Jordan centralizer is not in general a centralizer.

Bernes[2], Luh [10] and Kyuno[9] studied the structure of Γ-rings and obtained various generalizations of corresponding parts in ring theory.

Borut Zalar [15] worked on centralizers of semiprime rings and proved that Jordan centralizers and centralizers of this rings coincide. Joso Vukman[12, 13, 14] developed some remarkable results using centralizers on prime and semiprime rings.

Y.Ceven [3] worked on Jordan left derivations on completely prime Γ-rings. He investigated the existence of a nonzero Jordan left derivation on a completely prime Γ-ring that makes the

Γ-ring commutative with an assumption. With the same assumption, he showed that every Jordan left derivation on a completely prime Γ-ring is a left derivation on it.

In [4], M. F. Hoque and A.C. Paul have proved that every Jordan centralizer of a 2-torsion free semiprime Γ-ring is a centralizer. There they also gave an example of a Jordan centralizer which is not a centralizer.

In [5], M. F. Hoque and A.C. Paul have proved that if M is a 2-torsion free semiprime Γ-ring satisfying the assumption (A) and if $T : M \to M$ is an additive mapping such that $T(x\alpha y\beta x) = x\alpha T(y)\beta x$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, then T is a centralizer. Also, they have proved that T is a centralizer if M contains a multiplicative identity 1.

Our research works are inspired by the works of [1], [5], [7] and [8] and we obtain the results in Γ-rings with involution by assuming an assumption (A).

§2. The θ-Centralizers of Semiprime Gamma Rings with Involution

Definition 2.1 *Let M be a 2-torsion free semiprime Γ-ring and let θ be an endomorphism of M. An additive mapping $T : M \to M$ is a left(right) θ-centralizer if $T(x\alpha y) = T(x)\alpha\theta(y)(T(x\alpha y) = \theta(x)\alpha T(y))$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. If T is a left and a right θ-centralizer, then it is natural to call T a θ-centralizer.*

Definition 2.2 *Let M be a Γ-ring and let $a \in M$ and $\alpha \in \Gamma$ be fixed element. Let $\theta : M \to M$ be an endomorphism. Define a mapping $T : M \to M$ by $T(x)a\alpha\theta(x)$. Then it is clear that T is a left θ-centralizer. If $T(x) = \theta(x)\alpha a$ is defined, then T is a right θ-centralizer.*

Definition 2.3 *An additive mapping $T : M \to M$ is Jordan left(right) θ-centralizer if $T(x\alpha x) = T(x)\alpha\theta(x)(T(x\alpha x) = \theta(x)\alpha T(x))$ holds for all $x \in M$ and $\alpha \in \Gamma$.*

It is obvious that every left θ-centralizer is a Jordan left θ-centralizer but in general Jordan left θ-centralizer is not a left θ-centralizer [8, Example-2.1].

Definition 2.4 *Let M be a Γ-ring and let θ be an endomorphism on M. An additive mapping $T : M \to M$ is called a Jordan θ-centralizer if $T(x\alpha y + y\alpha x) = T(x)\alpha\theta(y) + \theta(y)\alpha T(x)$, for all $x, y \in M$ and $\alpha \in \Gamma$.*

It is clear that every θ-centralizer is a Jordan θ-centralizer but the converse is not in general a θ- centralizer [8, Example-2.2 and 2.3].

Definition 2.5 *An additive mapping $D : M \to M$ is called a (θ, θ)-derivation if $D(x\alpha y) = D(x)\alpha\theta(y) + \theta(x)\alpha D(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$ and is called a Jordan (θ, θ)-derivation if $D(x, x) = D(x)\alpha\theta(x) + \theta(x)\alpha D(x)$ holds for all $x \in M$ and $\alpha \in \Gamma$.*

We have given two examples in [8] which are ensure that a θ-centralizer and a Jordan θ-centralizer exist in Γ-ring.

Definition 2.6 *Let M be a Γ-ring. Then the mapping $I : M \to M$ is called an involution if*

(i) $II(a) = a$;
(ii) $I(a+b) = I(a) + I(b)$;
(iii) $I(a\alpha b) = I(b)\alpha I(a)$

for all $a, b \in M$ *and* $\alpha \in \Gamma$.

Example 2.1 Let R be a ring with involution I containing the unity element 1. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n_1.1 \\ n_2.1 \end{pmatrix} : n_1, n_2 \in Z \right\}$. Then M is a Γ-ring. We define an involution $I : M \to M$ by

$$I(a,b) = (I(a), I(b))$$

$$II(a,b) = (II(a), II(b)) = (a,b)$$

$$\begin{aligned} I((a,b)+(c,d)) &= I(a+c, b+d) \\ &= (I(a+c), I(b+d)) \\ &= (I(a)+I(c), I(b)+I(d)) \\ &= (I(a), I(b)) + (I(c), I(d)) \\ &= I(a,b) + I(c,d) \end{aligned}$$

Now

$$\begin{aligned} I\left((a,b)\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}(c,d)\right) &= I\left((an_1 + bn_2)(c,d)\right) \\ &= I(an_1c + bn_2c, an_1d + bn_2d) \\ &= (I(an_1c + bn_2c), I(an_1d + bn_2d)) \\ &= (I(an_1c) + I(bn_2c), I(an_1d) + I(bn_2d)) \\ &= (I(c)n_1I(a) + I(c)n_2I(b), I(d)n_1I(a) + I(d)n_2I(b)) \\ &= (I(c), I(d))\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}(I(a), I(b)) \\ &= I(c,d)\alpha I(a,b), \end{aligned}$$

where $\alpha = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$.

Definition 2.7 *Let M be a 2-torsion free semiprime Γ ring with involution I and let $\theta : M \to M$ be an endomorphism of M.An additive mapping $T : M \to M$ is called a left(right) Jordan $\theta-$centralizer with involutio if for all $x \in M$, $\alpha \in \Gamma$,*

$$T(x\alpha x) = T(x)\alpha\theta(I(x))(T(x\alpha x) = \theta(I(x))\alpha T(x)).$$

If T is both left and right Jordan $\theta-$centralizer of M with involution, then it is called Jordan $\theta-$centralizer of M with involution.

First, we need the following Lemmas, for proving our main results:

Lemma 2.1 *Suppose M is a semiprime Γ-ring satisfying the assumption (A). Suppose that the relation $x\alpha a\beta y + y\alpha a\beta z = 0$ holds for all $a \in M$, some $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then $(x + z)\alpha a\beta y = 0$ for all $a \in M$ and $\alpha, \beta \in \Gamma$.*

Lemma 2.2 *Suppose M is a 2-torsion free semiprime Γ-ring with involution I and satisfying the assumption (A). Let $T : M \to M$ be an additive mapping such that*

$$2T(x\alpha x) \quad = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$$

holds for all $x \in M$, $\alpha \in \Gamma$ and θ is an endomorphism on M. Then

$$\begin{aligned} 2T(x\alpha y + y\alpha x) \quad &= T(x)\alpha\theta(I(y)) + T(y)\alpha\theta(I(x)) \\ &+\theta(I(x))\alpha T(y) + \theta(I(y))\alpha T(x) \end{aligned}$$

for all $x, y \in M$.

Proof We have $2T(x\alpha x) = T(x)\alpha\theta(I(x))+\theta(I(x))\alpha T(x)$. By linearizing, the above relation becomes

$$\begin{aligned} 2T(x\alpha y + y\alpha x) \quad &= T(x)\alpha\theta(I(y)) + T(y)\alpha\theta(I(x)) \\ &+\theta(I(x))\alpha T(y) + \theta(I(y))\alpha T(x). \end{aligned} \tag{1}$$

This completes the proof. □

Lemma 2.3 *Suppose M is a 2-torsion free semiprime Γ-ring with involution I and satisfying the assumption (A). Let $T : M \to M$ be an additive mapping such that*

$$2T(x\alpha x) \quad = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$$

holds for all $x \in M$, $\alpha \in \Gamma$ and θ is an endomorphism on M. Then

$$\begin{aligned} 8T(x\alpha y\beta x) \quad &= T(x)\alpha\theta(I(x)\beta I(y) + 3I(y)\beta I(x)) + \theta(I(y)\beta I(x) \\ &+3I(x)\beta I(y))\alpha T(x) + 2\theta(I(x))\beta T(y)\alpha\theta(I(x)) \\ &-\theta(I(x)\alpha I(x)\beta T(y) - T(y)\beta\theta(I(x)\alpha I(x)) \end{aligned}$$

for all $x, y \in M$.

Proof Putting $2(x\beta y + y\beta x)$ for y in (1) and using Lemma 2.2, we get

$$\begin{aligned} &4T(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) \\ &\quad = T(x)\alpha\theta(2I(x)\beta I(y) + 3I(y)\beta I(x)) + \theta(3I(x)\beta I(y) \\ &\quad +2I(y)\beta I(x))\alpha T(x) + \theta(I(x))\alpha T(x)\beta\theta(I(y)) \\ &\quad +\theta(I(x)\alpha I(x)\beta T(y) + T(y)\beta\theta(I(x)\alpha I(x)) \\ &\quad +2\theta(I(x))\beta T(y)\alpha\theta(I(x)) + \theta(I(y)\beta T(x)\alpha\theta(I(x)) \end{aligned} \tag{2}$$

On the other hand

$$\begin{aligned}4T(x\alpha(x\beta y+y\beta x)+(x\beta y+y\beta x)\alpha x) &= 4T(x\alpha x\beta y+y\beta x\alpha x)\\ &+8T(x\alpha y\beta x)\end{aligned}$$

Now, using hypothesis, we obtain

$$\begin{aligned}&4T(x\alpha(x\beta y+y\beta x)+(x\beta y+y\beta x)\alpha x)\\ &\quad= T(x)\alpha\theta(I(x)\beta I(y))+\theta(I(y)\beta I(x))\alpha T(x)+\\ &\quad\theta(I(x))\alpha T(x)\beta\theta(I(y))+\theta(I(y))\beta T(x)\alpha\theta(I(x))\\ &\quad+2\theta(I(x)\alpha I(x))\beta T(y)+2T(y)\beta\theta(I(x)\alpha I(x))+8T(x\alpha y\beta x)\end{aligned}\tag{3}$$

Then from (2) and (3), we have

$$\begin{aligned}8T(x\alpha y\beta x) &= T(x)\alpha\theta(I(x)\beta I(y)+3I(y)\beta I(x))+\theta(I(y)\beta I(x)\\ &+3I(x)\beta I(y))\alpha T(x)+2\theta(I(x))\beta T(y)\alpha\theta(I(x))\\ &-\theta(I(x)\alpha I(x)\beta T(y)-T(y)\beta\theta(I(x)\alpha I(x))\end{aligned}\tag{4}$$

for all $x, y \in M$. This completes the proof. □

Lemma 2.4 *Suppose M is a 2-torsion free semiprime Γ-ring with involution I and satisfying the assumption (A). Let $T : M \to M$ be an additive mapping such that*

$$2T(x\alpha x) = T(x)\alpha\theta(I(x))+\theta(I(x))\alpha T(x)$$

holds for all $x \in M$, $\alpha \in \Gamma$ and θ is an endomorphism on M. Then

$$\begin{aligned}0 \;=\;& T(x)\alpha\theta(I(x)\gamma I(y)\beta I(x)-2I(y)\gamma I(x)\beta I(x)\\ &-2I(x)\beta(x)\gamma(y))+\theta(I(x)\gamma I(y)\beta I(x)\\ &-2I(y)\gamma I(x)\beta I(x)-2I(x)\beta I(x)\gamma I(y))\alpha T(x)\\ &+\theta(I(x)\alpha T(x)\beta\theta(I(x)\gamma I(y)+I(y)\gamma I(x))\\ &+\theta(I(x)\gamma I(y)+I(y)\gamma I(x))\beta T(x)\alpha\theta(I(x))\\ &+\theta(I(x)\alpha I(x))\gamma T(x)\beta\theta(I(y))\\ &+\theta(I(y))\beta T(x)\gamma\theta(I(x)I(x))\end{aligned}$$

for all $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$.

Proof Putting $8(x\beta y\gamma x)$ for y in (1) and using lemma (2.3), we obtain

$$\begin{aligned}16T(x\alpha x\beta y\gamma x+x\beta y\gamma x\alpha x) &= T(x)\alpha\theta(9I(x)\gamma I(y)\beta I(x)+3I(y)\gamma I(x)\beta I(x))\\ &+\theta(9I(x)\gamma I(y)\beta I(x)+3I(x)\beta I(x)\gamma I(y))\alpha T(x)\\ &+\theta(I(x))\alpha T(x)\beta\theta(I(x)\gamma I(y)+3I(y)\gamma I(x))\\ &+\theta(I(y)\gamma I(x)+3I(x)\gamma I(y))\beta T(x)\alpha\theta(I(x))\\ &-T(y)\gamma\theta(I(x)\beta I(x)\alpha I(x))\\ &+\theta(I(x)I(x))\gamma T(y)\beta\theta(I(x))\\ &+\theta(I(x))\gamma T(y)\beta\theta(I(x)\alpha I(x))\\ &-\theta(I(x)\alpha I(x)\beta I(x))\gamma T(y). \qquad (5)\end{aligned}$$

On the other hand using (4) and then after collecting some terms using Lemma 2.2, we obtain

$$\begin{aligned}16T(x\alpha x\beta y\gamma x+x\beta y\gamma x\alpha x) &= T(x)\alpha\theta(2I(x)\beta I(x)\gamma I(y)\\ &+5I(y)\beta I(x)\gamma I(x)+8I(x)\gamma I(y)\beta I(x))\\ &+\theta(2I(y)\beta I(x)\gamma I(x)+5I(x)\beta I(x)\gamma I(y)\\ &+8I(x)\gamma I(y)\beta I(x))\alpha T(x)\\ &+2\theta(I(x))\gamma T(x)\beta\theta(I(y)\alpha I(x))\\ &+2\theta(I(x)\gamma I(y))\beta T(x)\alpha\theta(I(x))\\ &+\theta(I(x)\alpha I(x))\gamma T(y)\beta\theta(I(x))\\ &+\theta(I(x))\gamma T(y)\beta\theta(I(x)\alpha I(x))\\ &-\theta(I(x)\alpha I(x))\gamma T(x)\beta\theta(I(y))\\ &-\theta(I(y))\beta T(x)\gamma\theta(I(x)\alpha I(x))\\ &-\theta(I(x)\beta I(x)\alpha I(x))\gamma T(y)\\ &-T(y)\gamma\theta(I(x)\alpha I(x)\beta I(x)). \qquad (6)\end{aligned}$$

By comparing (5)and (6), we get

$$\begin{aligned}0 \;=\;& T(x)\alpha\theta(I(x)\gamma I(y)\beta I(x)-2I(y)\gamma I(x)\beta I(x)-2I(x)\beta(x)\gamma(y))\\ &+\theta(I(x)\gamma I(y)\beta I(x)-2I(y)\gamma I(x)\beta I(x)-2I(x)\beta I(x)\gamma I(y))\alpha T(x)\\ &+\theta(I(x)\alpha T(x)\beta\theta(I(x)\gamma I(y)+I(y)\gamma I(x))+\theta(I(x)\gamma I(y)\\ &+I(x)\gamma I(y))\beta T(x)\alpha\theta(I(x))+\theta(I(x)\alpha I(x))\gamma T(x)\beta\theta(I(y))\\ &+\theta(I(y))\beta T(x)\gamma\theta(I(x)I(x)), \qquad (7)\end{aligned}$$

for $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. □

Lemma 2.5 *Suppose M is a 2-torsion free semiprime Γ-ring with involution I and satisfying the assumption (A). Let $T : M \to M$ be an additive mapping such that*

$$2T(x\alpha x) = T(x)\alpha\theta(I(x))+\theta(I(x))\alpha T(x)$$

holds for all $x \in M$,$\alpha \in \Gamma$ *and* θ *is an endomorphism on* M. *Then*

$$\begin{aligned} 0 &= \theta(I(x)\gamma I(y))\beta[\theta(I(x)\alpha I(x)), T(x)]_\alpha \\ &+2\theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]_\alpha \\ &+2\theta(I(y)\gamma I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha \\ &+\theta(I(y)\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \alpha\theta(I(x)) \\ &+\theta(I(y))\beta[\theta(I(x)), T(x)]_\alpha \gamma\theta(I(x)\alpha I(x)) \end{aligned}$$

for all $x, y \in M$ *and* $\alpha, \beta, \gamma \in \Gamma$.

Proof Replacing y by $x\alpha y$ in (7), we have

$$\begin{aligned} 0 \quad = \quad & T(x)\alpha\theta(I(x)\gamma I(y)\alpha I(x)\beta I(x) - 2I(y)\alpha I(x)\gamma I(x)\beta I(x) \\ & -2I(x)\beta(x)\gamma(y)\alpha I(x)) + \theta(I(x)\gamma I(y)\alpha I(x)\beta I(x) \\ & -2I(y)\alpha I(x)\gamma I(x)\beta I(x) - 2I(x)\beta I(x)\gamma I(y)\alpha I(x))\alpha T(x) \\ & +\theta(I(x))\alpha T(x)\beta\theta(I(x)\gamma I(y)\alpha I(x) + I(y)\alpha I(x)\gamma I(x)) \\ & +\theta(I(x)\gamma I(y)\alpha I(x) + I(y)\alpha I(x)\gamma I(x))\beta T(x)\alpha\theta(I(x)) \\ & +\theta(I(x)\alpha I(x))\gamma T(x)\beta\theta(I(y)\alpha I(x)) \\ & +\theta(I(y)\alpha I(x))\beta T(x)\gamma\theta(I(x)\alpha I(x)). \end{aligned} \tag{8}$$

Right multiplication of (7) by $\theta(I(x))$, we get

$$\begin{aligned} 0 \quad = \quad & T(x)\alpha\theta(I(x)\gamma I(y)\beta I(x)\alpha(I(x)) - 2I(y)\gamma I(x)\beta I(x)\alpha I(x) \\ & -2I(x)\beta(x)\gamma(y)\alpha I(x)) + \theta(I(x)\gamma I(y)\beta I(x) \\ & -2I(y)\gamma I(x)\beta I(x) - 2I(x)\beta I(x)\gamma I(y))\alpha T(x)\alpha\theta(I(x)) \\ & +\theta(I(x)\alpha T(x)\beta\theta(I(x)\gamma I(y) \\ & +I(y)\gamma I(x))\alpha\theta(I(x)) + \theta(I(x)\gamma I(y) \\ & +I(y)\gamma I(x))\beta T(x)\alpha\theta(I(x)\alpha I(x)) \\ & +\theta(I(x)\alpha I(x))\gamma T(x)\beta\theta(I(y)\alpha I(x)) \\ & +\theta(I(y))\beta T(x)\gamma\theta(I(x)\alpha I(x)\alpha I(x)). \end{aligned} \tag{9}$$

Subtracting (9) from (8) and using assumption(A), we have

$$\begin{aligned} 0 &= \theta(I(x)\alpha I(y)\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \\ &+2\theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]_\alpha \\ &+2\theta(I(y)\gamma I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha \\ &+\theta(I(x)\gamma I(y))\beta[\theta(I(x)), T(x)]_\alpha \alpha\theta(I(x)) \\ &+\theta(I(y)\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \alpha\theta(I(x)) \\ &+\theta(I(y))\beta[\theta(I(x)), T(x)]_\alpha \gamma\theta(I(x)\alpha I(x)) \end{aligned}$$

Now combining first and fourth term together this relation reduces as,

$$\begin{aligned} 0 &= \theta(I(x)\gamma I(y))\beta[\theta(I(x)\alpha I(x)), T(x)]_\alpha \\ &+2\theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x), \theta(I(x))]_\alpha \\ &+2\theta(I(y)\gamma I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha \\ &+\theta(I(y)\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \alpha\theta(I(x)) \\ &+\theta(I(y))\beta[\theta(I(x)), T(x)]_\alpha \gamma\theta(I(x)\alpha I(x)) \end{aligned} \tag{10}$$

for all $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. □

Lemma 2.6 *Suppose M is a 2-torsion free semiprime Γ-ring with involution I and satisfying the assumption (A). Let $T : M \to M$ be an additive mapping such that*

$$2T(x\alpha x) = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$$

holds for all $x \in M$, $\alpha \in \Gamma$ and θ is an endomorphism on M. Then

$$\begin{aligned} 0 &= [T(x), \theta(I(x))]_\alpha \gamma\theta(y)\beta[T(x), \theta(I(x)\alpha I(x))]_\alpha \\ &-2[T(x), \theta(I(x)\alpha I(x))]_\alpha \theta(y)\beta[T(x), \theta(I(x))]_\alpha \end{aligned}$$

for all $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$.

Proof First replacing y by I(y) in (10), we have

$$\begin{aligned} 0 &= \theta(I(x)\gamma y)\beta[\theta(I(x)\alpha I(x)), T(x)]_\alpha \\ &+2\theta(I(x)\alpha I(x)\gamma y)\beta[T(x), \theta(I(x))]_\alpha \\ &+2\theta(y\gamma I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha \\ &+\theta(y\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \alpha\theta(I(x)) \\ &+\theta(y)\beta[\theta(I(x)), T(x)]_\alpha \gamma\theta(I(x)\alpha I(x)) \end{aligned}$$

Now putting $\theta(y) = T(x)\alpha\theta(I(y))$

$$\begin{aligned} 0 &= \theta(I(x))\gamma T(x)\alpha\theta(I(y))\beta[\theta(I(x)\alpha I(x)), T(x)]_\alpha \\ &+2\theta(I(x)\alpha I(x))\gamma T(x)\alpha\theta(I(y))\beta[T(x), \theta(I(x))]_\alpha \\ &+2T(x)\alpha\theta(I(y)\gamma I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha \\ &+T(x)\alpha\theta(I(y)\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \alpha\theta(I(x)) \\ &+T(x)\alpha\theta(I(y))\beta[\theta(I(x)), T(x)]_\alpha \gamma\theta(I(x)\alpha I(x)). \end{aligned} \tag{11}$$

Left multiplication of (11) by $T(x)\alpha$, we get

$$\begin{aligned} 0 &= T(x)\alpha\theta(I(x)\gamma I(y))\beta[\theta(I(x)\alpha I(x)), T(x)]_\alpha \\ &+2T(x)\alpha\theta(I(x)\alpha I(x))\gamma I(y))\beta[T(x), \theta(I(x))]_\alpha \\ &+2T(x)\alpha\theta(I(y)\gamma I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha \\ &+T(x)\alpha\theta(I(y)\gamma I(x))\beta[\theta(I(x)), T(x)]_\alpha \alpha\theta(I(x)) \\ &+T(x)\alpha\theta(I(y))\beta[\theta(I(x)), T(x)]_\alpha \gamma\theta(I(x)\alpha I(x)). \end{aligned} \tag{12}$$

Subtracting (12)from (11), we arrive at

$$\begin{aligned} 0 &= [T(x),\theta(I(x))]_\alpha\gamma\theta(I(y))\beta[T(x),\theta(I(x)I(x))]_\alpha - \\ &\quad 2[T(x),\theta(I(x)\alpha I(x))]_\alpha\gamma\theta(I(y))\beta[T(x),\theta(I(x))]_\alpha \end{aligned}$$

Replacing y by I(y), we get

$$\begin{aligned} 0 &= [T(x),\theta(I(x))]_\alpha\gamma\theta(y)\beta[T(x),\theta(I(x)\alpha I(x))]_\alpha \\ &\quad -2[T(x),\theta(I(x)\alpha I(x))]_\alpha\gamma\theta(y)\beta[T(x),\theta(I(x))]_\alpha \end{aligned} \tag{13}$$

for all $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. □

Lemma 2.7 *Suppose M is a 2-torsion free semiprime Γ-ring with involution I and satisfying the assumption (A). Let $T : M \to M$ be an additive mapping such that*

$$2T(x\alpha x) \quad = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$$

holds for all $x \in M$, $\alpha \in \Gamma$ and θ is an endomorphism on M. Then

$$[T(x),\theta(I(x)\alpha I(x))]_\alpha \quad = 0$$

for all $x \in M$.

Proof Now replacing $\theta(y)$ by r and taking $a = [T(x),\theta(I(x))]_\alpha$ $b = [T(x),\theta(I(x)\alpha I(x))]_\alpha$ and $c = -2[T(x),\theta(I(x)\alpha I(x))]_\alpha$ in (13), we get $a\gamma r\beta b + c\gamma r\beta a = 0$ for all $r \in M$. Hence using Lemma 2.1, we obtain that $(c + b)\beta r\gamma a = 0$, which implies that

$$[T(x),\theta(I(x)\alpha I(x))]_\alpha\beta r\gamma[T(x),\theta(I(x))]_\alpha = 0.$$

Using this relation, we arrive at

$$\begin{aligned} 0 &= [T(x),\theta(I(x)\alpha I(x))]_\alpha\beta r\gamma(\theta(I(x))\alpha[T(x),\theta(I(x))]_\alpha \\ &\quad +[T(x),\theta(I(x))]_\alpha\alpha\theta(I(x))) \end{aligned}$$

We therefore have

$$[T(x),\theta(I(x)\alpha I(x))]_\alpha\beta r\gamma[T(x),\theta(I(x)\alpha I(x))]_\alpha \quad = 0$$

for all $r \in M$.

Hence by semiprimeness of M, we have

$$[T(x),\theta(I(x)\alpha I(x))]_\alpha \quad = 0, \tag{14}$$

for all $x \in M$. □

Theorem 2.1 *Suppose M is a 2-torsion free semiprime Γ-ring with involution I and satisfying the assumption (A). Let $T : M \to M$ be an additive mapping such that*

$$2T(x\alpha x \quad = T(x)\alpha\theta(I(x)) + \theta(I(x))\alpha T(x)$$

holds for all $x \in M$, $\alpha \in \Gamma$ and θ is an endomorphism on M. Then T is a Jordan $\theta-$centralizer.

Proof Linearizing the relation given in Lemma-2.7, we get

$$\begin{aligned} 0 &= [T(x), \theta(I(y)\alpha I(y))]_\alpha + [T(y), \theta(I(x)\alpha I(x))]_\alpha + [T(x), \theta(I(x)\alpha I(y) \\ &\quad + I(y)\alpha I(x))]_\alpha + [T(y), \theta(I(x)\alpha I(y) + I(y)\alpha I(x))]_\alpha \end{aligned}$$

Putting in above relation $-x$ for x and comparing the relation so obtained with the above relation and by 2-torsion freeness of M,

$$[T(x), \theta(I(x)\alpha I(y) + I(y)\alpha I(x))]_\alpha + [T(y), \theta(I(x)\alpha I(x))]_\alpha = 0 \tag{15}$$

Replacing $2(x\beta y + y\beta x)$ for y and using Lemma 2.7, we obtain

$$\begin{aligned} 0 &= [T(x), \theta(I(x)\alpha 2I(x\beta y + y\beta x) + 2I(x\beta y + y\beta x)\alpha I(x))]_\alpha \\ &\quad + [T(x\beta y + y\beta x), \theta(I(x)\alpha I(x))]_\alpha \\ &\quad = 2[T(x), \theta(I(x)\alpha I(x)\beta I(y) + I(y)\beta I(x)\alpha I(x) + 2I(x)\alpha I(y)\beta I(x))]_\alpha + \\ &\quad [T(x)\beta\theta(I(y)) + \theta(I(x))\beta T(y) + T(y)\beta\theta(I(x)) \\ &\quad + \theta(I(y))\beta T(x), \theta(I(x)\alpha I(x))]_\alpha \end{aligned}$$

Thus we have

$$\begin{aligned} 0 &= 2\theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]_\alpha + 2[T(x), \theta(I(y))]_\alpha\beta\theta(I(x)\alpha I(x)) \\ &\quad 4[T(x), \theta(I(x)\alpha I(y)\beta I(x))]_\alpha + T(x)\beta[\theta(I(y)), \theta(I(x)\alpha I(x))]_\alpha \\ &\quad + [\theta(I(y)), \theta(I(x)\alpha I(x))]_\alpha\beta T(x) + \theta(I(x))\beta[T(y), \theta(I(x)\alpha I(x))]_\alpha + \\ &\quad [T(y), \theta(I(x)\alpha I(x))]_\alpha\beta\theta(I(x). \end{aligned} \tag{16}$$

Hence in particular, we find that

$$\begin{aligned} 0 &= \theta(I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha + [T(x), \theta(I(x))]_\alpha\beta\theta(I(x)\alpha I(x)) \\ &\quad + 2[T(x), \theta(I(x)\beta I(x)\alpha I(x))]_\alpha \end{aligned}$$

In view of Lemma-2.7, this reduces to

$$\begin{aligned} 0 &= \theta(I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha + 3[T(x), \theta(I(x))]_\alpha\beta \\ &\quad \theta(I(x)\alpha I(x)). \end{aligned} \tag{17}$$

According to Lemma 2.7, we get

$$[T(x), \theta(I(x))]_\alpha\beta\theta(I(x)) + \theta(I(x))\beta[T(x), \theta(I(x))]_\alpha = 0$$

Hence using the later relation, we find that

$$\theta(I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha = [T(x), \theta(I(x))]_\alpha\beta\theta(I(x)\alpha I(x))$$

Further using this replacement in (17), we have

$$[T(x), \theta(I(x))]_\alpha\beta\theta(I(x)\alpha I(x)) = 0 \tag{18}$$

for all $x \in M$ and $\alpha, \beta \in \Gamma$.

Similarly

$$\theta(I(x)\alpha I(x))\beta[T(x), \theta(I(x))]_\alpha = 0 \tag{19}$$

for all $x \in M$ and $\alpha, \beta \in \Gamma$.

We also have

$$\theta(I(x))\beta[T(x), \theta(I(x))]_\alpha \alpha\theta(I(x)) = 0 \tag{20}$$

for all $x \in M$ and $\alpha, \beta \in \Gamma$.

From (15) we have

$$[T(x), \theta(I(x)\alpha I(y) + I(y)\alpha I(x))]_\alpha \quad = -[T(y), \theta(I(x)\alpha I(x))]_\alpha$$

and combining this fact with (16),we arrive at

$$\begin{aligned}
0 \quad = \quad & 2\theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]_\alpha + 2[T(x), \theta(I(y))]_\alpha\beta\theta(I(x)\alpha I(x)) \\
& +4[T(x), \theta(I(x)\beta I(y)\alpha I(x))]_\alpha + T(x)\beta[\theta(I(y)), \theta(I(x)\alpha I(x))]_\alpha \\
& +[\theta(I(y)), \theta(I(x)\alpha I(x))]_\alpha\beta T(x) - \theta(I(x))\beta[T(x), \theta(I(x)\alpha I(y) \\
& +I(y)\alpha I(x))]_\alpha - [T(x), \theta(I(x)\alpha I(y) + I(y)\alpha I(x))]_\alpha\beta\theta(I(x)) \\
& = 2\theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]_\alpha + 2[T(x), \theta(I(y))]_\alpha\beta\theta(I(x)\alpha I(x)) \\
& +4[T(x), \theta(I(x))]_\alpha\beta\theta(I(y)\alpha I(x)) + 4\theta(I(x))\beta[T(x), \theta(I(y))]_\alpha\alpha\theta(I(x)) \\
& +4\theta(I(x)\alpha I(y))\beta[T(x), \theta(I(x))]_\alpha + T(x)\beta[\theta(I(y)), \theta(I(x)\alpha I(x))]_\alpha \\
& +[\theta(I(y), \theta(I(x)\alpha I(x))]_\alpha\beta T(x) - \theta(I(x))\beta[T(x), \theta(I(x))]_\alpha\alpha\theta(I(y)) \\
& -\theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]_\alpha - \theta(I(x))\beta[T(x), \theta(I(y))]_\alpha\alpha\theta(I(x)) \\
& -\theta(I(x)\alpha I(y))\beta[T(x), \theta(I(x))]_\alpha - [T(x), \theta(I(x))]_\alpha\beta\theta(I(y)\alpha I(x)) \\
& -\theta(I(x))\beta[T(x), \theta(I(y))]_\alpha\alpha\theta(I(x)) - [T(x), \theta(I(y))]_\alpha\beta\theta(I(x)\alpha I(x)) \\
& -\theta(I(y))\beta[T(x), \theta(I(x))]_\alpha\alpha\theta(I(x))
\end{aligned}$$

Hence, we have

$$\begin{aligned}
0 \quad = \quad & \theta(I(x)\alpha I(x))\beta[T(x), \theta(I(y))]_\alpha + [T(x), \theta(I(y))]_\beta\theta(I(x)\alpha I(x)) \\
& +3[T(x), \theta(I(x))]_\alpha\beta\theta(I(y)\alpha I(x)) + 3\theta(I(x)\alpha I(y))\beta[T(x), \theta(I(x))]_\alpha \\
& +2\theta(I(x))\beta[T(x), \theta(I(y))]_\alpha\alpha\theta(I(x)) + T(x)\beta[\theta(I(y)), \theta(I(x)\alpha I(x))]_\alpha \\
& +[\theta(I(y)), \theta(I(x)\alpha I(x))]_\alpha\beta T(x) - \theta(I(x))\beta[T(x), \theta(I(x))]_\alpha\alpha\theta(I(y)) \\
& -\theta(I(y))\beta[T(x), \theta(I(x))]_\alpha\alpha\theta(I(x)).
\end{aligned} \tag{21}$$

Replacing y by $x\gamma y$, we have

$$\begin{aligned}
0 \quad = \quad & \theta(I(x)\alpha I(x))\beta[T(x),\theta(I(y)\gamma I(x))]_\alpha \\
& +[T(x),\theta(I(y)\gamma I(x))]_\alpha\beta\theta(I(x)\alpha I(x)) \\
& +3[T(x),\theta(I(x))]_\alpha\beta\theta(I(y)\gamma I(x)\alpha I(x)) \\
& +3\theta(I(x)\alpha I(y)\gamma I(x))\beta[T(x),\theta(I(x))]_\alpha \\
& +2\theta(I(x))\beta[T(x),\theta(I(y)\gamma I(x))]_\alpha\alpha\theta(I(x)) \\
& +T(x)\beta[\theta(I(y)\gamma I(x)),\theta(I(x)\alpha I(x))]_\alpha \\
& +[\theta(I(y)\gamma I(x)),\theta(I(x)\alpha I(x))]_\alpha\beta T(x) \\
& -\theta(I(x))\beta[T(x),\theta(I(x))]_\alpha\alpha\theta(I(y)\gamma I(x)) \\
& -\theta(I(y)\gamma I(x))\beta[T(x),\theta(I(x))]_\alpha\alpha\theta(I(x))
\end{aligned}$$

This can be written as (also using assumption (A))

$$\begin{aligned}
0 \; & = \theta(I(x)\alpha I(x))\beta[T(x),\theta(I(y))]_\alpha\gamma\theta(I(x)) \\
& +\theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x),\theta(I(x))]_\alpha \\
& +[T(x),\theta(I(y))]_\alpha\beta\theta(I(x)\alpha I(x)\gamma I(x)) \\
& +\theta(I(y))\beta[T(x),\theta(I(x))]_\alpha\gamma\theta(I(x)\alpha I(x)) \\
& +3[T(x),\theta(I(x))]_\alpha\beta\theta(I(y)\gamma I(x)\alpha I(x)) \\
& +3\theta(I(x)\alpha I(y)\gamma I(x))\beta[T(x),\theta(I(x))]_\alpha \\
& +2\theta(I(x))\beta[T(x),\theta(I(y))]_\alpha\gamma\theta(I(x)\alpha I(x)) \\
& +2\theta(I(x)\alpha I(y))\beta[T(x),\theta(I(x))]_\alpha\gamma\theta(I(x)) \\
& +T(x)\beta[\theta(I(y)),\theta(I(x)\alpha I(x))]_\alpha\gamma\theta(I(x)) \\
& +[\theta(I(y)),\theta(I(x)\alpha I(x))]_\alpha\gamma\theta(I(x))\beta T(x) \\
& -\theta(I(x))\beta[T(x),\theta(I(x))]_\alpha\gamma\theta(I(y)I(x)) \\
& -\theta(I(y)\alpha I(x))\beta[T(x),\theta(I(x))]_\alpha\gamma\theta(I(x))
\end{aligned}$$

In view of (18) and (20), we have

$$\begin{aligned}
0 \quad = \quad & \theta(I(x)\alpha I(x))\beta[T(x),\theta(I(y))]_\alpha\gamma\theta(I(x)) \\
& +\theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x),\theta(I(x))]_\alpha \\
& +[T(x),\theta(I(y))]_\alpha\beta\theta(I(x)\alpha I(x)\gamma I(x)) \\
& +3[T(x),\theta(I(x))]_\alpha\beta\theta(I(y)\gamma I(x)\alpha I(x)) \\
& +3\theta(I(x)\alpha I(y)\gamma I(x))\beta[T(x),\theta(I(x))]_\alpha \\
& +2\theta(I(x))\beta[T(x),\theta(I(y))]_\alpha\gamma\theta(I(x)\alpha I(x)) \\
& +2\theta(I(x)\alpha I(y))\beta[T(x),\theta(I(x))]_\alpha\gamma\theta(I(x)) \\
& +T(x)\beta[\theta(I(y)),\theta(I(x)\alpha I(x))]_\alpha\gamma\theta(I(x)) \\
& +[\theta(I(y)),\theta(I(x)\alpha I(x))]_\alpha\gamma\theta(I(x))\beta T(x) \\
& -\theta(I(x))\beta[T(x),\theta(I(x))]_\alpha\gamma\theta(I(y)\alpha I(x)). \qquad (22)
\end{aligned}$$

Right multiplication of (21) by $\gamma\theta(I(x))$ gives

$$\begin{aligned} 0 \;=\;& \theta(I(x)\alpha I(x))\beta[T(x),\theta(I(y))]_\alpha\gamma\theta(I(x)) \\ &+[T(x),\theta(I(y))]_\alpha\beta\theta(I(x)\alpha I(x)\gamma I(x)) \\ &+3[T(x),\theta(I(x))]_\alpha\beta\theta(I(y)\alpha I(x)\gamma I(x)) \\ &+3\theta(I(x)\alpha I(y))\beta[T(x),\theta(I(x))]_\alpha\gamma\theta(I(x)) \\ &+2\theta(I(x))\beta[T(x),\theta(I(y))]_\alpha\alpha\theta(I(x)\gamma I(x)) \\ &+T(x)\beta[\theta(I(y)),\theta(I(x)\alpha I(x))]_\alpha\gamma\theta(I(x)) \\ &+[\theta(I(y)),\theta(I(x)\alpha I(x))]_\alpha\beta T(x)\gamma\theta(I(x)) \\ &-\theta(I(x))\beta[T(x),\theta(I(x))]_\alpha\alpha\theta(I(y)\gamma I(x)) \end{aligned} \tag{23}$$

Subtracting (23) from (22), we get (also we using assumption (A))

$$\begin{aligned} 0 \;=\;& \theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x),\theta(I(x))]_\alpha \\ &+3\theta(I(x)\gamma I(y))\beta[\theta(I(x)),[T(x),\theta(I(x))]_\alpha]_\alpha \\ &+2\theta(I(x)\gamma I(y))\beta[T(x),\theta(I(x))]_\alpha\alpha\theta(I(x)) \\ &+[\theta(I(y),\theta(I(x)\gamma I(x))]_\alpha\beta[\theta(I(x)),T(x)]_\alpha \end{aligned}$$

Further in view of (19) this yields

$$\begin{aligned} 0 \;=\;& 2\theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x),\theta(I(x))]_\alpha \\ &+3\theta(I(x)\gamma I(y)\alpha I(x))\beta[T(x),\theta(I(x))]_\alpha \\ &-\theta(I(x)\gamma I(y))\beta[T(x),\theta(I(x))]_\alpha\alpha\theta(I(x)) \end{aligned}$$

In view of Lemma 2.7, the above relation yields that

$$\begin{aligned} 0 \;=\;& \theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x),\theta(I(x))]_\alpha \\ &+2\theta(I(x)\gamma I(y)\alpha I(x))\beta[T(x),\theta(I(x))]_\alpha \end{aligned} \tag{24}$$

Further application of Lemma 2.7, (18), (19), (20) together with Lemma 2.5 yields that

$$0=\theta(I(x)\alpha I(x)\gamma I(y))\beta[T(x),\theta(I(x))]_\alpha. \tag{25}$$

Hence combining (24) and (25) and by 2-torsion freeness of M, we have

$$\theta(I(x)\gamma I(y)\alpha I(x))\beta[T(x),\theta(I(x))]_\alpha \quad =0$$

Now put $y=I(y)$, we have

$$\theta(I(x)\gamma y\alpha I(x))\beta[T(x),\theta(I(x))]_\alpha \quad =0$$

Now replacing $\theta(y)$ by $[T(x),\theta(I(x))]_\alpha\alpha\theta(y)$ in the later expression, we have (also using assumption (A))

$$\theta(I(x))\alpha[T(x),\theta(I(x))]_\alpha\gamma\theta(y)\beta\theta(I(x))\alpha[T(x),\theta(I(x))]_\alpha \quad =0$$

As θ is an endomorphism, the semiprimeness of M gives

$$\theta(I(x))\alpha[T(x),\theta(I(x))]_\alpha = 0 \tag{26}$$

and hence in view of Lemma 2.7, we can write

$$[T(x),\theta(I(x))]_\alpha\alpha\theta(I(x)) = 0. \tag{27}$$

Linearizing (26) and using (27), we get

$$\begin{aligned} 0 &= \theta(I(x))\alpha[T(x),\theta(I(y))]_\alpha + \theta(I(x))\alpha[T(y),\theta(I(x))]_\alpha \\ &+\theta(I(x))\alpha[T(y),\theta(I(y))]_\alpha + \theta(I(y))\alpha[T(x),\theta(I(x))]_\alpha \\ &+\theta(I(y))\alpha[T(x),\theta(I(y))]_\alpha + \theta(I(y))\alpha[T(y),\theta(I(x))]_\alpha \end{aligned}$$

Putting in the above relation $-x$ for x and comparing the relation so obtained with the above we get,

$$\begin{aligned} 0 &= \theta(I(x))\alpha[T(x),\theta(I(y))]_\alpha + \theta(I(x))\alpha[T(y),\theta(I(x))]_\alpha \\ &+\theta(I(y))\alpha[T(x),\theta(I(x))]_\alpha \end{aligned}$$

Now, multiply the above relation by $[T(x),\theta(I(x))]_\alpha\alpha$ from left and use (27), we have

$$[T(x),\theta(I(x))]_\alpha\alpha\theta(I(y))\alpha[T(x),\theta(I(x))]_\alpha \quad = 0$$

This follows that,

$$[T(x),\theta(I(x))]_\alpha = 0. \tag{28}$$

Combining (28) with our hypothesis, we get

$$T(x\alpha x) \quad = T(x)\alpha\theta(I(x)) forall x \in M$$

and

$$T(x\alpha x) \quad = \theta(I(x))\alpha T(x) forall x \in M$$

This means that T is a left and right Jordan $\theta-$centralizer. This complete the proof of our theorem. ◻

Corollary 2.1 *Suppose that M be a 2-torsion free semiprime Γ-ring with involution I and satisfying the assumption(A). If $T : M \to M$ be an additive mapping such that*

$$2T(x\alpha x) \quad = T(x)\alpha I(x) + I(x)\alpha T(x)$$

holds for all $x \in M$ and $\alpha \in \Gamma$, then T is a Jordan centralizer with involution I.

References

[1] M.Ashraf and M.R. Mozumder, On Jordan $\alpha-^\star$ centralizers in semiprime rings with involution, *Int. J. Contemp. Math. Sciences*, Vol.7, 2012, No.23, 1103 - 1112.

[2] W.E.Bernes, On the Γ-rings of Nobusawa, *Pacific J.Math.*, 18(1966), 411-422.

[3] Y.Ceven, Jordan left derivations on completely prime gamma rings, C.U.Fen-Edebiyat Fakultesi,Fen Bilimleri Dergisi (2002)Cilt 23 Sayi 2.

[4] M.F.Hoque and A.C.Paul, On centralizers of semiprime gamma rings, *International Mathematical Forum*, Vol.6(2011), No.13, 627-638.

[5] M.F.Hoque and A.C.Paul, Centralizers on semiprime gamma rings, *Italian J. of Pure and Applied Mathematics*, Vol. 30(2013), 289-302.

[6] M.F.Hoque and A.C.Paul, An equation related to centralizers in semiprime gamma rings, *Annals of Pure and Applied Mathematics*, Vol.1 (2012), 84-90.

[7] M.F.Hoque, H.O.Roshid and A.C.Paul, An equation related to θ-centralizers in semiprime gamma rings, *International J. Math. Combin.*, Vol. 4 (2012), 17-26.

[8] M.F.Hoque and A.C.Paul, The θ-centralizers of semiprime gamma rings, *Research Journal of Applied Sciences, Engineering and Technology*, 6(22), (2013), 4129-4137.

[9] S.Kyuno, On prime gamma ring, *Pacific J.Math.*,75(1978), 185-190.

[10] L.Luh, On the theory of simple gamma rings, *Michigan Math.J.*, 16(1969), 65-75.

[11] N.Nobusawa, On the generalization of the ring theory, *Osaka J. Math.*,1(1964), 81-89.

[12] J.Vukman and I. Kosi-Ulbl, On centralizers of semiprime rings with involution, *Studia Sci. Math. Hungarica*, 43,1(2006), 61-67.

[13] J.Vukman, An identity related to centralizers in semiprime rings, *Comment. Math. Univ. Carolinae*, 40, 3(1999), 447-456.

[14] J.Vukman, Centralizers on semiprime rings, *Comment. Math. Univ. Carolinae*, 42, 2(2001), 237-245.

[15] B.Zalar, On centralizers of semiprime rings, *Comment.Math. Univ. Carolinae*, 32(1991), 609-614.